« Missing data in physics » workshop

Nice, May 11th. & 12th. 2015

Influence of irregular nodes spacing in noisy rational interpolation

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The function $f(z)$ is a rational interpolant to $\varphi(z)$

if

1. $f(z)$ is a rational function

2. $f(z)$ and $\varphi(z)$ are equal at certain points or nodes $x_1, x_2, \ldots x_M$

In other words $f(z)$ is known everywhere, while $\varphi(z)$ is known only at the M nodes .

The quantities $\varphi(x_j)$ are the data; they may come from experiment.
The Data Analysis amounts to determining the zeros and poles of $f(z)$.

In the following, we will stick with the $\varphi(z)$ itself rational case.

Introducing stochasticity

As the data $\varphi(x_j)$ may come from experiment, they may be affected by random uncertainties.

The data are now the random variables D_j
which may be modeled e.g. as follows

$$D_j = (1+\varepsilon r_j)\varphi(x_j)$$

ε is here to monitor the expansion in the random part and the r_j 's are independent random variables, e.g. gaussian with zero mean and variance unity.

Noisy rational interpolation

The rational interpolation conditions become : $f(z)$ rational and random, with the conditions

$$f(x_j) = D_j = (1+\varepsilon r_j)\varphi(x_j)$$

The Data Analysis amounts to determining the statistics of the zeros and poles of $f(z)$.

A toy model which exhibits the relevant phenomenon

$$\varphi(z) \equiv \Phi \qquad f(z) = \Phi \frac{p_0 + p_1 z}{q_0 + q_1 z} \qquad x_1 = -1 \quad x_2 = e \quad x_3 = +1 \qquad M = 3 \qquad e \in [-1,+1]$$

If $e = 0$, the nodes are regularly spaced.
If $e \neq 0$, there is some asymmetry and one may conversely speak of missing nodes or missing data.

The resulting rational random interpolating function $f(z)$ reads

$$f(z) = \Phi \frac{K(z) + \varepsilon P(z)}{K(z)}$$

where $K(z) and P(z)$ are degree one random polynomials with

$$K(z) = r_3 - r_1 + e(r_3 - 2r_2 + r_1) - [(1-e)r_1 - 2r_2 + (1+e)r_3]z$$

The random polynomial $K(z)$ is called the « Froissart Polynomial » (FP)

The last task is to examine the probability distribution function of the root of FP,
which in turn governs the statistics of the zero and the pole of $f(z)$ to the lowest order in ε .

The pdf of the root of the FP in the toy model

It can be derived exactly under the assumptions of the toy model ; it is a Cauchy-Lorentz law, where the only parameter left free is e ,the position of the middle node, namely

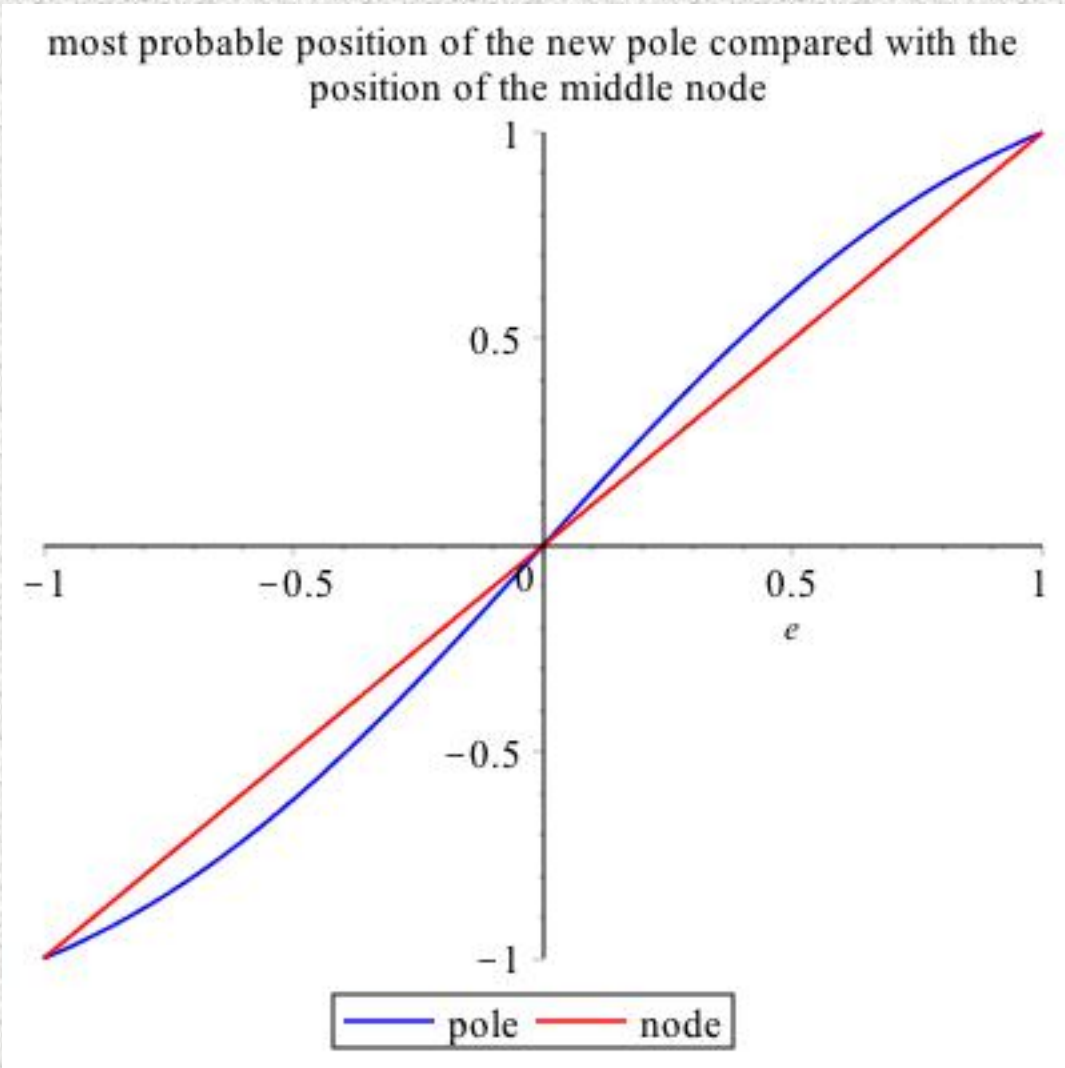
$$\rho(z) = \frac{1}{\pi} \frac{\Lambda}{\Lambda^2 + (z - z_c)^2}$$

with an extension $$\Lambda = 3^{1/2} \frac{(1 - e^2)}{(3 + e^2)}$$

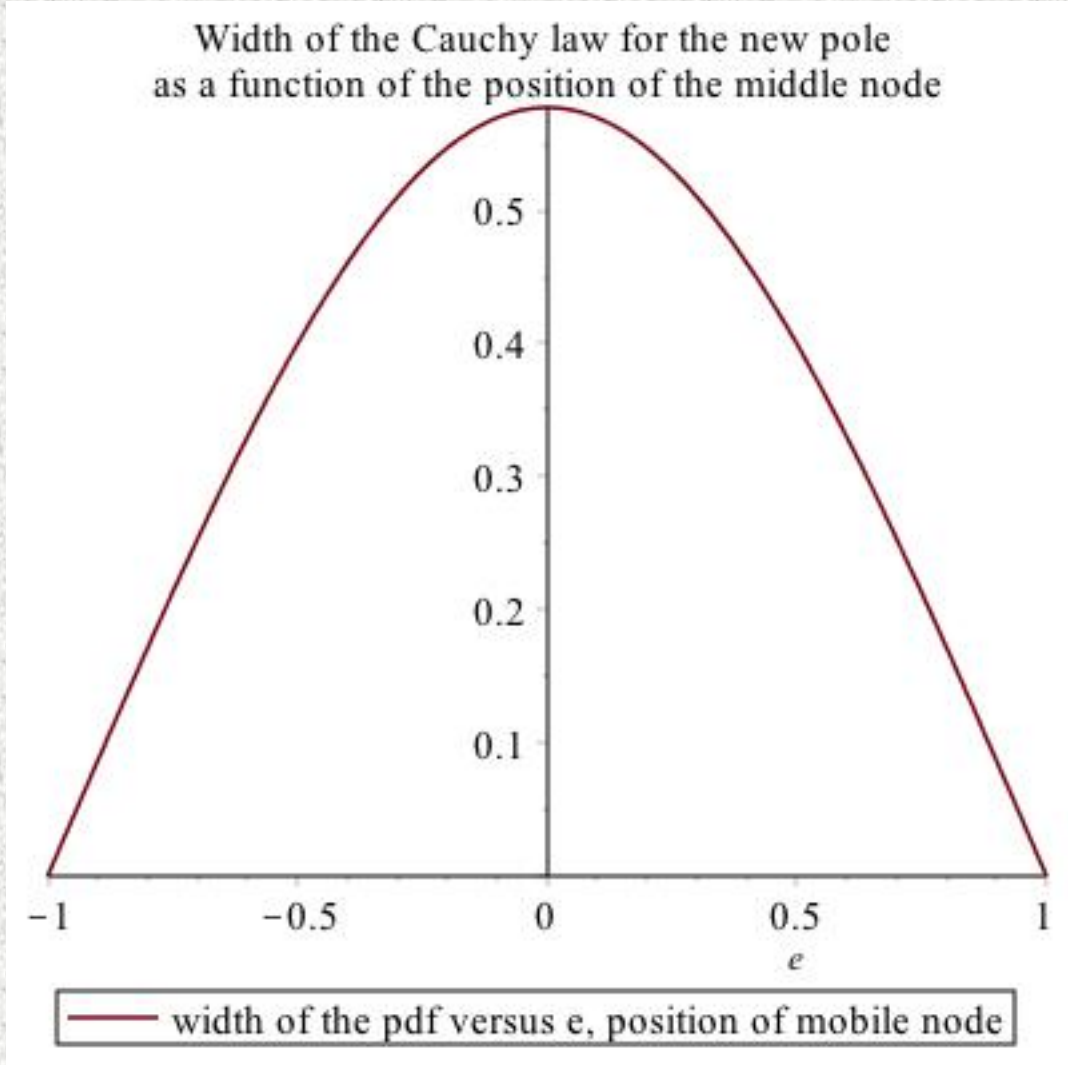
and a center $$z_c = 4 \frac{e}{3 + e^2}$$

The hereafter analysis of these three functions for varying e and z completes our study.

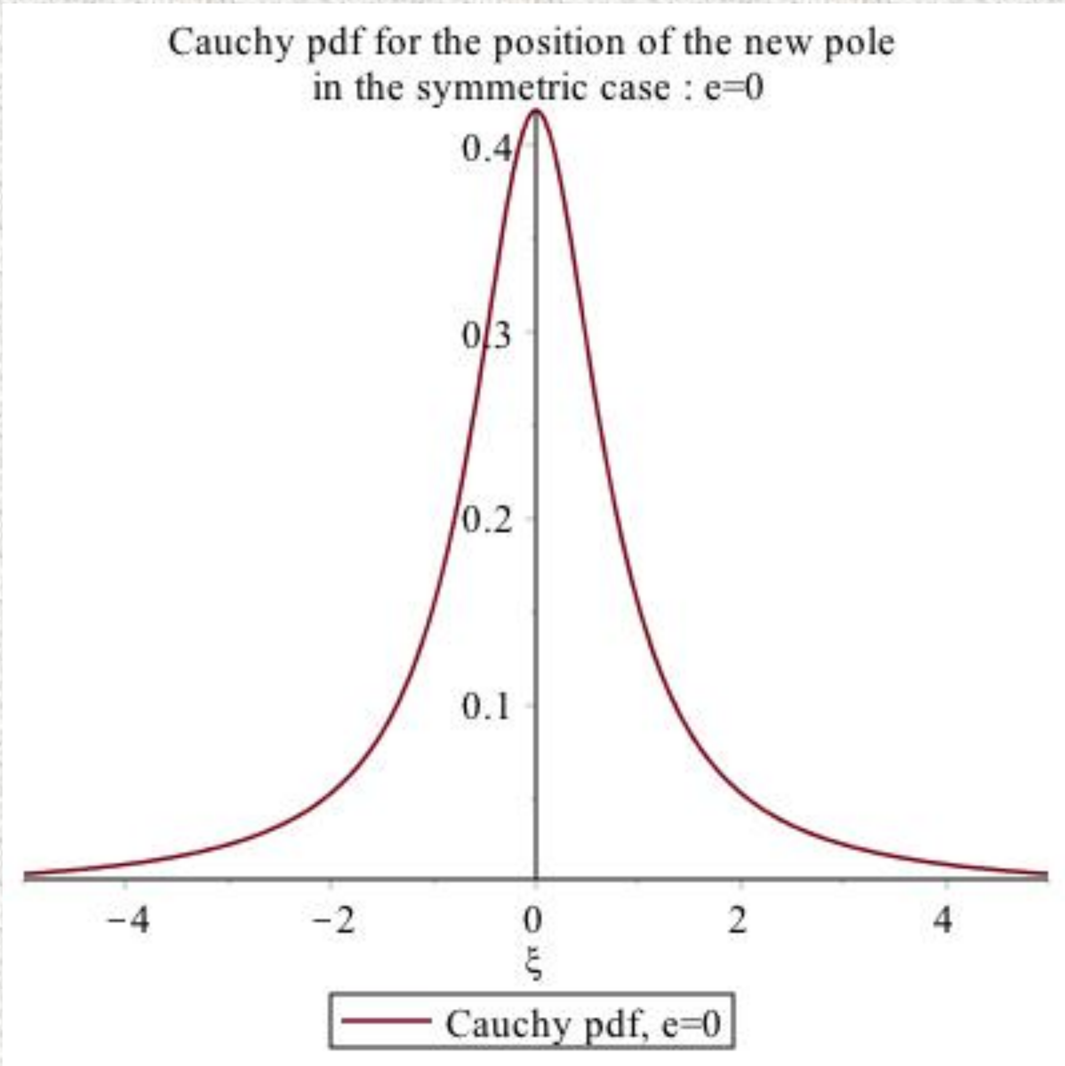
plot([4e/(e^2+3), e], e = -1 .. 1, color = [blue, red]);



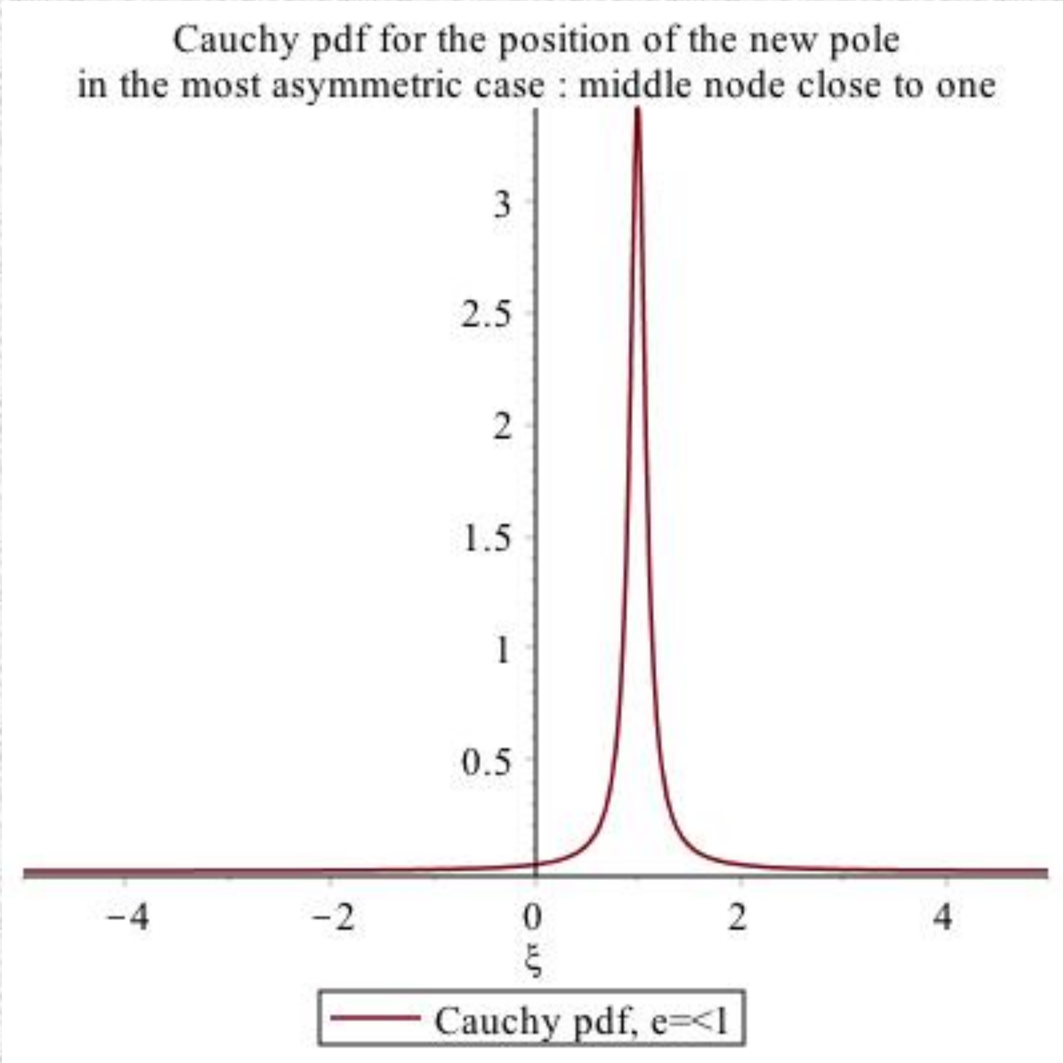
plot(sqrt(3)(-e^2+1)/(e^2+3), e = -1 .. 1)



plot(subs(e = 0, rho(xi)), xi = -5 .. 5)



plot(subs(e = .99, rho(xi)), xi = -5 .. 5)



Influence of irregular nodes spacing in noisy rational interpolation

To the lowest order in the noise amplitude, a certain 'Froissart' Polynomial (FP) governs the statistics of the additional zeros and poles in stochastically perturbed rational interpolation.

The FP is actually a random polynomial and there is an interplay between the noise, the pattern of the interpolation nodes and the statistical pattern of the roots of the FP.